CALCULUS II
ECCENTRICITY

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1. Goal

We wish to view all conic sections as a continuously varying family of curves, defined by a single equation which contains variables x and y, as well as *parameters*, which are constants which we allow to vary. That the family varies continuously means that small changes in the parameters cause small changes in the curves.

To simplify the ideas and the computations, we wish to fix the center of the conic sections to be the origin. Unfortunately, this precludes including parabolas in the family, and instead produces the degenerate case of two parallel lines where one would suspect a parabola to belong.

2. Review

The standard equation of an ellipse centered at the origin with horizontal focal axis is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,}$$

where $a \geq b$. The *vertices* are the x-intercepts $(\pm a, 0)$, the *covertices* are the y-intercepts $(0, \pm b)$, and the *foci* are $(\pm c, 0)$, where

$$c^2 = a^2 - b^2.$$

If $b > a$, the vertices and foci are on the y-axis and $c^2 = b^2 - a^2$.

The standard equation of a hyperbola centered at the origin with vertical focal axis is

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

The *vertices* are the x-intercepts $(\pm a, 0)$, the *covertices* are $(0, \pm b)$, and the *foci* are $(\pm c, 0)$, where

$$c^2 = a^2 + b^2.$$

The asymptotes are the lines $y = \pm \frac{b}{a}$.

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3. A Family Parameterized by Time

We begin by considering the equation

$$\frac{x^2}{a^2} + t\frac{y^2}{b^2} = 1.$$

For a fixed a and t, this is a conic section. The parameter a represents the size of the curve, and we will leave a fixed. Think of the parameter t as time, so that as time passes, the equation produces curves in a moving picture. All of the curves have x-intercepts at $(\pm a, 0)$.

For $t = 1$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1,$$

which is the equation of a circle of radius a centered at the origin.

For $t > 1$, as t increases, the curve flattens into an ellipse. For example, at $t = 4$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{(a/2)^2} = 1,$$

whose y-intercepts are $(0, \pm\frac{a}{2})$. As t approaches ∞, the curve approaches the horizontal line segment $[-a, a]$ on the x-axis.

For $0 < t < 1$, as t decreases from 1 to 0, the curve stretches vertically into an ellipse whose focal axis is the y-axis. For example, at $t = \frac{1}{4}$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{(2a)^2} = 1,$$

whose y-intercepts are $(0, \pm 2a)$. As t approaches 0, the ellipse becomes indefinitely tall.

For $t = 0$, the equation is

$$\frac{x^2}{a^2} = 1, \qquad \text{or} \qquad x = \pm a.$$

The graph of this equation is a pair of vertical lines.

For $t < 0$, the y^2 term is negative and we obtain the equation of a hyperbola. For example, at $t = -1$, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1,$$

which is a hyperbola centered at the origin with horizontal focal axis and asymptotes $y = \pm x$. More generally, if $m > 0$ and $t = -m^2$, the asymptotes have slope $\frac{1}{m}$.

4. Eccentricity

The *eccentricity* of an ellipse or a hyperbola is

$$\boxed{e = \frac{c}{a} = \frac{\text{distance between foci}}{\text{distance between vertices}}.}$$

Thus $c = ae$.

For an ellipse with $b > a$, we can compute b^2 in terms of a and e as

$$c^2 = b^2 - a^2 \quad \Rightarrow \quad b^2 = a^2 - c^2 = a^2 - a^2e^2 = a^2(1 - e^2).$$

For a hyperbola, we have

$$c^2 = b^2 + a^2 \quad \Rightarrow \quad b^2 = c^2 - a^2 = a^2e^2 - a^2 = a^2(1 - e^2).$$

In either case, the equation of the conic centered at the origin with size a and eccentricity e is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

where ellipses have a vertical focal axis. This produces a family of curves parameterized by a and e, related to the previous discussion by substituting $t = \frac{1}{1-e^2}$.

5. Generalized Parabolas

Select a point P and line L not containing P. Let Q be another point on the plane, and let $d(Q, P)$ be the distance from Q to P, and let $d(Q, L)$ be the distance from Q to L. Then, by definition, Q is on the *parabola* with focus P and directrix L if and only if $d(Q, P) = d(Q, L)$. Dividing both sides by $d(Q, L)$ gives $\frac{d(Q,P)}{d(Q,L)} = 1$.

We generalize this as follows.

Select any point P, a line L not containing p, and a positive real number e. Consider the locus of the equation

$$\frac{d(Q, P)}{d(Q, L)} = e.$$

We call P the *focus*, L the *directrix*, and e the *eccentricity*, of this locus. If $e = 1$, this is a parabola. What type of locus does this equation create if $e \neq 1$? To understand this, let us change the scale and shift. We will rescale back at the end.

Let $a = \frac{ed(P,L)}{|e^2-1|}$; this is the *size* of the locus. Shrink the plane by a factor of a; for simplicity consider P, Q, and L in this shrunken plane without relabeling them. The distance between the focus and the directrix is now

$$d(P, L) = \frac{|e^2 - 1|}{e} = |e - \frac{1}{e}|.$$

By a rigid motion of the plane, assume that $P = (e, 0)$ and $L : x = e$. Then

$$\begin{aligned}\frac{d(Q,p)}{d(Q,L)} = 1 &\Leftrightarrow d(Q,P)^2 = e^2 d(Q,L)^2 \\ &\Leftrightarrow (x-e)^2 + y^2 = e^2(x - \frac{1}{e})^2 \\ &\Leftrightarrow x^2 - 2ex + e^2 + y^2 = e^2x^2 - 2ex + 1 \\ &\Leftrightarrow (1-e^2)x^2 + y^2 = 1 - e^2 \\ &\Leftrightarrow x^2 + \frac{y^2}{1-e^2} = 1.\end{aligned}$$

To scale back to the original size, we need to stretch the plane by a factor of a. This is done by making the substitutions

$$x \rightsquigarrow \frac{x}{a} \quad \text{and} \quad y \rightsquigarrow \frac{y}{a},$$

which produces the following equation, which is an alternate form of the equation of the locus of the generalized parabola with focus $(ae, 0)$ and directrix $x = \frac{a}{e}$:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.}$$

This is exactly the equation we had for ellipses and hyperbolas parameterized by a and e!

If $0 < e < 1$, this is the equation of an ellipse; set $b^2 = a^2(1-e^2)$. Then $c^2 = a^2 - b^2 = a^2(1-(1-e^2)) = a^2e^2$, so $c = ae$, and $e = \frac{c}{a}$ is the eccentricity.

If $e > 1$, this is the equation of a hyperbola; set $b^2 = a^2(e^2-1)$, so $c^2 = a^2+b^2 = a^2(e^2-1+1) = a^2e^2$, and again, $e = \frac{c}{a}$.

6. Conclusion

We have shown how ellipses and hyperbolas are generalizations of parabolas, that their foci act as generalizations of a parabolic focus, that they also have directrices, and that they are parameterized by their eccentricities in this generalization.

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